Exploring Black Hole Discharge in Massive Electrodynamics

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the Honors Bachelor's Degree

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Abstract

In the study of black holes, there exists a remarkable set of theorems known as no-hair theorems. These theorems state that under certain general conditions, black holes can be described completely by just three parameters: their mass, charge, and angular momentum. In particular, when an electric charge falls into a black hole, if the photon is massive, the profile of the electric field outside of the black hole rapidly approaches the source-free limit, so that one cannot know the fate of the charge at late times. Here, we consider black hole discharge for a massive vector field. We postulate the discharge rate for charged black holes in massive electrodynamics and numerically confirm the rate of discharge in the limit of small photon mass.

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1 Introduction

Black holes are astrophysical objects corresponding to point mass solutions of Einstein's equations of general relativity. When solving the Einstein equations (in general a collection of sixteen coupled differential equations) for certain boundary conditions, a set of uniqueness theorems holds. By solving the Einstein equations, we mean that for a given arrangement of matter and energy in space, we determine the corresponding metric and therefore characterize the curvature of spacetime. Suppose we require that our black hole solutions be stationary, asymptotically flat (i.e. reduce to the flat-space metric far from the black hole), and nonsingular outside the event horizon. If we consider a theory in which electromagnetism is the only long-range nongravitational field present, then our black hole solutions can be described completely by three parameters: their mass, (electric) charge, and angular momentum [1].

These uniqueness theorems, otherwise known as no-hair theorems, distinguish black holes from more standard matter distributions like stars and planets. Whereas one might need a large or possibly infinite set of parameters to describe the structure of a planet, a black hole can be described entirely by a much smaller set of parameters due to the singularity that defines it and the corresponding presence of an event horizon. Here, we analyze the case of massive vector hairs and show that in massive electrodynamics, black holes must discharge. Furthermore, we compute the discharge rate and numerically confirm a fitting formula originally found by Mirbabayi and Gruzinov [2].

2 Solving the Proca equation

The fundamental equation of massive electrodynamics is the Proca equation,

$$\partial_\nu(\sqrt{-g}g^{\nu\alpha}g^{\mu\beta}F_{\alpha\beta}) + m^2\sqrt{-g}A^\mu = 4\pi\sqrt{-g}J^\mu. \tag{1}$$

The Proca equation governs the behavior of the electric and magnetic fields of black holes and charged stars in massive electrodynamics, analogous to the Maxwell equations in the ordinary theory of electromagnetism. Here, A^μ is the four-vector potential, g is the determinant of the metric $g_{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, and J^μ is the four-current density.

We will show that the Proca equation can be written as an eigenvalue problem for the electric field, a system which is in principle numerically integrable.

2.1 Static charged solutions

Let us consider spherically symmetric solutions to the Proca equation where the metric is diagonal in (t, r, θ, ϕ) coordinates. Following Mirbabayi and Gruzinov, we note that by the spherical symmetry, the only nonvanishing component of $F_{\mu\nu}$ is F_{tr}. But by definition, $F_{tr} = \partial_t A_r - \partial_r A_t = E$.

Now the $\mu = r$ component of equation 1 reads

$$\partial_t(\sqrt{-g}g^{tt}g^{rr}F_{tr}) + m^2\sqrt{-g}g^{rr}A_r = 4\pi\sqrt{-g}J^r, \tag{2}$$

and similarly for $\mu = t$,

$$\partial_r(\sqrt{-g}g^{rr}g^{tt}F_{rt}) + m^2\sqrt{-g}g^{tt}A_t = 4\pi\sqrt{-g}J^t. \tag{3}$$

We begin by considering static charged solutions ($\partial_t = 0$). As the charge approaches the event horizon, its proper time moves increasingly slowly relative to an external observer, and so the Proca equation becomes source-free at late times, $J^\mu \to 0$ [3]. Now because $J^r = 0$ and $\partial_t = 0$, Eqn. 2 forces $A_r = 0$ for all static solutions (given that $m \neq 0$). But consider now the quantity $g^{\mu\nu}A_\mu A_\nu$, an observable in Proca theory [2]. Since $A_r = 0$, this product reduces to $g^{tt}A_t^2$. If $A_t = 0$ at the horizon and at $r \to +\infty$, then Eqn. 3 tells us that A_t vanishes identically, $A_t(r) = 0$ (this trivially solves the equation, and by uniqueness must be the only solution). On the other hand, if $A_t \neq 0$ then g^{tt} diverges at the horizon, and so no static charged solutions can exist if the photon is massive. We conclude that in massive electrodynamics, black holes must discharge.

2.2 Time-dependent charged solutions

Next we consider the time-dependent solutions (so that A_r need not be zero). Let us assume that the charge is small so that the Schwarzschild metric applies:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{4}$$

with $d\Omega = d\theta^2 + \sin^2 d\phi^2$ as usual and r_g the Schwarzschild radius of the black hole. With the Schwarzschild metric, we find that

$$\det g = -\left(1 - \frac{r_g}{r}\right)\left(1 - \frac{r_g}{r}\right)^{-1} (r^2)(r^2 \sin^2\theta) = -r^4 \sin^2\theta$$

so that

$$\sqrt{-g} = r^2 \sin\theta.$$

We can rewrite equations 2 and 3 as a single equation for the electric field as follows. Substituting in the explicit elements of the inverse metric, we take $J^\mu = 0$ (the source-free limit) and take the time derivative of Eqn. 2:

$$\begin{aligned} 0 &= \partial_t\left(\partial_t(\sqrt{-g}g^{tt}g^{rr}F_{tr}) + m^2\sqrt{-g}g^{rr}A_r\right) \\ &= \partial_t\left(\partial_t(r^2\sin\theta(-1)E) + m^2(r^2\sin\theta)(-\left(1 - \frac{r_g}{r}\right))A_r\right) \\ &= \partial_t^2 E + m^2\left(1 - \frac{r_g}{r}\right)\partial_t A_r. \end{aligned} \tag{5}$$

Next we rewrite Eqn. 3,

$$\begin{aligned} 0 &= \partial_r(\sqrt{-g}g^{rr}g^{tt}F_{rt}) + m^2\sqrt{-g}g^{tt}A_t \\ &= \partial_r((r^2\sin\theta)(-1)(-E)) + m^2(r^2\sin\theta)\left(1 - \frac{r_g}{r}\right)^{-1} A_t \\ &= \left(1 - \frac{r_g}{r}\right)\frac{1}{r^2}\partial_r(r^2 E) + m^2 A_t \end{aligned}$$

and take the radial derivative of the resulting expression,

$$\begin{aligned} 0 &= \partial_r\left(\left(1 - \frac{r_g}{r}\right)\frac{1}{r^2}\partial_r(r^2 E)\right) + m^2\partial_r A_t \\ &= \left(1 - \frac{r_g}{r}\right)\partial_r\left(\left(1 - \frac{r_g}{r}\right)\frac{1}{r^2}\partial_r(r^2 E)\right) \\ &\quad + m^2\left(1 - \frac{r_g}{r}\right)\partial_r A_t. \end{aligned} \tag{6}$$

Subtracting Eqn. 6 from Eqn. 5 (and using the fact that $\partial_t A_r - \partial_r A_t = E$), we recover the expression from Mirbabayi for the physical electric field,

$$\partial_t^2 E - \left(1 - \frac{r_g}{r}\right)\partial_r\left(\left(1 - \frac{r_g}{r}\right)\frac{1}{r^2}\partial_r(r^2 E)\right) + m^2\left(1 - \frac{r_g}{r}\right)E = 0. \tag{7}$$

Moreover, by defining the tortoise coordinate

$$\rho \equiv r + r_g \ln(r/r_g - 1)$$

and defining $\Psi \equiv rE$, one can rewrite this expression in the following form:

$$(\partial_t^2 - \partial_\rho^2 + V)\Psi = 0, \tag{8}$$

where V is the effective potential given by

$$V = \left(1 - \frac{r_g}{r}\right)\left(\frac{2}{r^2} - \frac{3r_g}{r^3} + m^2\right). \tag{9}$$

We show explicitly that this is the case by multiplying Eqn. 7 through by r and rewriting the r derivatives with the chain rule. First note that Eqn. 7 can be written as

$$\begin{aligned} 0 &= \partial_t^2\Psi - r\left(1-\frac{r_g}{r}\right)\partial_r\left(\left(1-\frac{r_g}{r}\right)\frac{1}{r^2}\partial_r(r\Psi)\right) + m^2\left(1-\frac{r_g}{r}\right)\Psi \\ &= \partial_t^2\Psi - r\left(1-\frac{r_g}{r}\right)\partial_r\left(\left(1-\frac{r_g}{r}\right)\frac{1}{r^2}(r\partial_r\Psi+\Psi)\right) + m^2\left(1-\frac{r_g}{r}\right)\Psi \end{aligned}$$

By the chain rule, $\partial_r = \frac{\partial\rho}{\partial r}\partial_\rho = \left(1+\frac{r_g}{r}(1-r_g/r)^{-1}\right)\partial_\rho$, or equivalently $\left(1-\frac{r_g}{r}\right)\partial_r = \partial_\rho$. We find that

$$\begin{aligned} 0 &= \partial_t^2\Psi - r\left(1-\frac{r_g}{r}\right)\partial_r\left(\left(1-\frac{r_g}{r}\right)\frac{1}{r^2}(r\partial_r\Psi+\Psi)\right) + m^2\left(1-\frac{r_g}{r}\right)\Psi \\ &= \partial_t^2\Psi - r\left(1-\frac{r_g}{r}\right)\partial_r\left(\frac{1}{r}\partial_\rho\Psi + \left(1-\frac{r_g}{r}\right)\frac{1}{r^2}\Psi)\right) + m^2\left(1-\frac{r_g}{r}\right)\Psi \\ &= \partial_t^2\Psi - \partial_\rho^2\Psi + \frac{1}{r}\left(1-\frac{r_g}{r}\right)\partial_\rho\Psi - r\left(1-\frac{r_g}{r}\right)\partial_r\left(\left(1-\frac{r_g}{r}\right)\frac{1}{r^2}\Psi)\right) + m^2\left(1-\frac{r_g}{r}\right)\Psi \\ &= \partial_t^2\Psi - \partial_\rho^2\Psi + \frac{1}{r}\left(1-\frac{r_g}{r}\right)\partial_\rho\Psi - r\left(1-\frac{r_g}{r}\right)\partial_r\left(\left(\frac{1}{r^2}-\frac{r_g}{r^3}\right)\Psi)\right) + m^2\left(1-\frac{r_g}{r}\right)\Psi \\ &= \partial_t^2\Psi - \partial_\rho^2\Psi + \frac{1}{r}\left(1-\frac{r_g}{r}\right)\partial_\rho\Psi + \left(1-\frac{r_g}{r}\right)\left(\frac{2}{r^2}-\frac{3r_g}{r^3}\right)\Psi - \frac{1}{r}\left(1-\frac{r_g}{r}\right)\partial_\rho\Psi + m^2\left(1-\frac{r_g}{r}\right)\Psi \\ &= (\partial_t^2 - \partial_\rho^2 + V)\Psi, \end{aligned}$$

which is exactly Eqn. 8 with the effective potential as given in Eqn. 9.

We then postulate that the field can be written as the product of a spatial profile $R(\rho)$ and an exponential time decay, so that

$$\Psi(\rho,t) = e^{-\gamma t}R(\rho). \tag{10}$$

This is a reasonable guess to make because of how the photon mass term appears in the Proca equation in the form of Eqn. 1. We see that the mass term sources a "screening charge" carried by the field itself, so that schematically,

$$J^\mu \sim m^2 A^\mu. \tag{11}$$

However, since $J^\mu \sim \frac{dq}{dt}$ and $A^\mu \propto q$, we find that

$$\frac{dq}{dt} \propto m^2 q \implies q(t) \sim e^{-m^2 t}. \tag{12}$$

This demonstrates that an exponential time decay is a natural guess for the behavior of the electric field, and moreover that the rate γ should be proportional to the photon mass squared. More carefully, by dimensional analysis we find that to a first approximation, the decay constant should be $\gamma \approx m^2 r_g$. This also agrees with our intution that for an ordinary massless photon, a black hole can maintain a nontrivial profile without discharging for arbitrarily long periods of time (i.e. as $m \to 0$, the characteristic time of decay $1/\gamma$ becomes infinite).

Now if this ansatz holds, then we can rewrite our problem as the following eigenvalue problem [2]:

$$\left(-\left(\frac{d}{d\rho}\right)^2 + \gamma^2 + V\right)\Psi = 0, \tag{13}$$

$$\frac{d\Psi}{d\rho} = -\gamma\Psi, \rho \to -\infty, \tag{14}$$

$$\Psi \to 0, \rho \to +\infty. \tag{15}$$

Here, the first boundary condition comes from the fact that $V \to 0$ near the horizon as $\rho \to -\infty$, and the second boundary condition is due to the requirement that the physical electric field vanish far away from the black hole. This eigenvalue problem can be solved numerically, and we present here some results from the numerical integration.

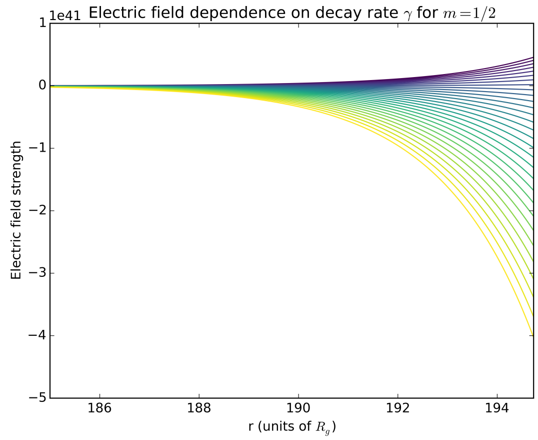

Figure 1: The limiting behavior of the electric field for a photon mass $m = 1/2$ and a Schwarzschild radius r_g normalized to 1. Here, darker colors represent smaller values of the time decay parameter γ and lighter colors represent larger values. The determined value of γ was approximately 0.1620, in comparison with the formula prediction of $\gamma = 0.1667$ (a 2.8% error).

3 Numerical integration

Using the SciPy Python libraries, we integrate Eqn. 13 to solve for Ψ as a function of ρ (or equivalently E as a function of r) for many values of γ. Our solutions turn out to vary smoothly with γ, and we can determine the physical solution by looking at the limiting behavior of the electric field E (i.e. does E go to zero as $r \to +\infty$?).

One example of this approach is seen in Figure 1. In this figure, we observe that all computed solutions of the system given by Eqns. 13 to 15 appear exponential far from the event horizon, growing to either positive or negative infinity. However, we may note that because these solutions also seem to depend smoothly on γ, the desired solution which decays to zero as $r \to \infty$ should be found somewhere between the smallest positive solution and the smallest negative solution. This provides us a means of placing bounds on the physical value of the time decay rate γ.

Let us recall that the two relevant energy scales in this problem are set by the photon mass m and the Schwarzschild radius r_G (which appears in the potential term V, for example). By repeating this process for many combinations of m and r_G, we can determine the dependence of γ on these two parameters, and some of these numerical scans appear in Figure 2.

For comparison, we have plotted a fitting formula determined by Mirbabayi et al along with our computed data. Mirbabayi and Gruzinov found that for $mr_g < 0.5$, the decay constant is well-approximated by the formula [2]

$$\gamma \approx \frac{m^2 r_g}{1 + mr_g}, \tag{16}$$

and this formula is represented as dashed lines in Figure 2. With our computations, we confirm this result to within 3% accuracy.

However, as the photon mass m gets larger, our numerical codes indicate an unexpected divergence. Specifically, as m approaches $1/r_G$, the decay rate γ diverges to $+\infty$, and this result holds over several values of r_G. This result is particularly surprising because a priori, one might expect that the decay rate becomes linear in m or otherwise levels off so that the exponential decay is still a good description of the time evolution of the system. At this point, it is unclear whether this effect is an indication that the exponential assumption breaks down earlier than expected, or whether there are unknown numerical errors producing a spurious result.

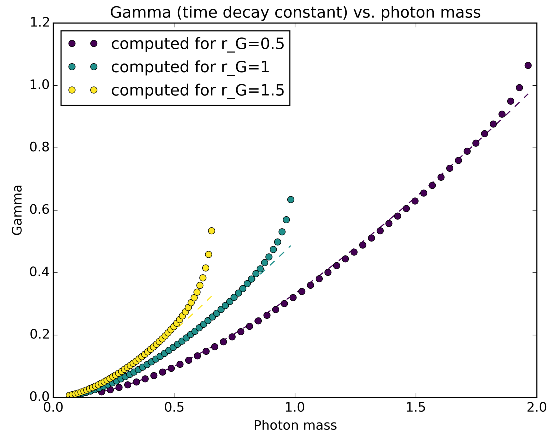

Figure 2: Several numerical computations of the electric field decay rate γ as a function of the photon mass m for three different values of r_g. Dashed lines indicate the fitting formula from Mirbabayi et al. Note the divergence for each plot as m approaches $1/r_g$.

4 Conclusions

Future work will focus on implementing more general and more robust numerical solving methods to probe this large-mass limit more carefully. We have indeed confirmed the fitting formula from Mirbabayi and Gruzinov, but our large-mass computations are suggestive of some new behavior. If the results of the current procedure are supported by other methods, this will be a strong indication that the model of exponential time decay of the electric field should be revised for large photon mass.

Let us also note that the discharge process we have described is completely independent of astrophysical black hole discharge, where black hole charge is rapidly neutralized by interactions with nearby matter [1]. Instead, it arises from the mass coupling term in the Proca equation, which precludes the existence of static charged solutions. However, in extreme cases such as the black hole mergers which are likely to be observed by detectors like LIGO and LISA, the discharge process due to the Proca mass coupling may be more significant, and the gravitational waves from the merger would be altered during the ringdown.

One could then imagine using the gravitational wave signals from black hole mergers to place an upper limit on the photon mass, for example, or to probe for the presence of another massive vector particle corresponding to electromagnetism. However, until more sensitive measurements can be made and this work is extended to better understand the large-mass limit, such procedures will remain highly speculative, and we therefore defer further discussion of them to future work.

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